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7102-0027-MU000

ANALYSIS OF BALLISTIC MISSILE PERFORMANCE

Part I. Basic Theory

April 21, 1961

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ABSTRACT

The analysis of ballistic missile performance involves the determination of targets for which the effect of propellant depletion is statistically acceptable. Therefore Q_p , the probability of avoiding propellant depletion prior to normal guidance shut down, plays a fundamental role. It is convenient to introduce a propellant reserve function $\mathcal{M}_p$ and express Q_p as the probability that $\mathcal{M}_p \geq 0$. This probability is determined in terms of the statistics of system parameters by assuming a linear expansion of the propellant reserve function over a region corresponding to dispersion for a particular launch-site/target combination. By approximating the probability distribution function for $\mathcal{M}_p$ as an equivalent normal one, an explicit solution can be obtained in terms of the expected value, $\overline{\mathcal{M}}_p$, and the standard deviation σ_p .

Alternately, a range function can be utilized to obtain the same result. When considering the target-range at constant probability it is useful to define range-exchange coefficients. Methods are discussed for utilizing such exchange coefficients to adjust previously obtained performance results to account for changes in system parameters. Also, some additional approximations are considered. The results are not essentially new but an attempt at a more complete and rigorous presentation is attempted.

1. RANGE PERFORMANCE AS STATISTICAL TARGET CAPABILITY

The quantitative definition of range performance for a ballistic missile requires a more detailed concept than simply that of "firing the missile as far as it will go". To arrive at a suitable definition, let us consider a particular launch site of interest, along with the corresponding set of operationally-shaped trajectories for some specified initial azimuth angle. There will then exist an associated locus of targets on the earth starting at some minimum allowable target-range and extending away from the launch-site. To define a corresponding "maximum target-range" we must establish the extreme target point out along this locus to which we can actually target the system without deterioration of weapon effectiveness to some unacceptable level. The associated question of targeting capability therefore involves first a definition of system effectiveness for any launch-site /target combination (necessarily in statistical terms), and second the specification of "acceptability" as determined by an appraisal of the exigencies of the military/economic situation. The present discussion will deal with only the technical problem of determining system effectiveness for all targets of interest from any particular launch-site.

For any selected target a fundamental measure of system performance is kill probability, which is directly related to the impact statistics. For a sufficiently high probability of achieving proper guidance shutdown without running out of propellant, the impact statistics depends only on the accuracy capability of the guidance system. However, as targets are selected at greater and greater ranges, the statistics for propellant depletion has a significant effect, so as to increase impact dispersion and consequently decrease kill probability. Thus kill probability varies with range to the target. For targets at medium ranges this probability is at an almost constant value corresponding to the statistics of guidance accuracy, and then decreases sharply as targets at greater ranges are considered and propellant depletion becomes significant. The "maximum target-range" then corresponds to the target for which this deterioration in kill probability due to propellant depletion has reached some acceptable (small) limit value, with targets at greater ranges corresponding to lower kill probabilities.

The effect of propellant depletion statistics upon impact statistics, and consequently upon kill probability, for a given launch-site for various target-ranges is embodied in the quantity $\mathcal{P}_p$, the probability of avoiding propellant depletion prior to normal guidance shut down (for a non-malfunctioning missile). A direct relationship can be established so that for a given launch-site and initial azimuth, we can consider the probability $\mathcal{P}_p$ as a convenient measure of system "range performance" that is equivalent to a measure of "effectiveness" for the target being considered. The target at maximum range thus corresponds to the value of $\mathcal{P}_p$ that has been separately determined as being consistent with a specified allowable deterioration in kill probability due to propellant depletion.

To introduce $\mathcal{P}_p$ in mathematical terms we consider it as the conditional probability for any launch-site/target combination corresponding to appropriate design operating conditions such as wind, atmosphere, etc. Let ϕ_L, λ_L, H_L denote the geographic latitude, geodetic longitude, and geodetic altitude respectively for the launch-site and ϕ_T, λ_T, H_T similarly for the target. Then we can represent $\mathcal{P}_p$ by a function as follows:

$$\mathcal{P}_p = \mathcal{P}_p\{\phi_L, \Delta\phi, \Delta\lambda, H_L, H_T\} \tag{1.1}$$

where

$$\Delta\phi \equiv \phi_T - \phi_L \tag{1.2}$$

$$\Delta\lambda \equiv \lambda_T - \lambda_L \tag{1.3}$$

As brought out in the previous discussion, it is useful to introduce a range quantity corresponding to some appropriate measure of separation between points on the earth. One possibility for this is a quantity proportional to the central angle for the two points (ϕ_L, λ_L) and (ϕ_T, λ_T) on the reference ellipsoid. We could also consider the geocentric plane through these two points and define range as the distance along its elliptical intersection with the reference ellipsoid. If we let R_T denote the range from launch-site to target, then we can express it as a function by

$$R_T = R_T(\phi_L, \Delta\phi, \Delta\lambda) \tag{1.4}$$

Rather than utilize the geodetic coordinate quantities $\Delta\phi$, $\Delta\lambda$ to represent the target location for a given launch site, it is more convenient for performance analysis to introduce some locus of targets on the reference ellipsoid and utilize the target-range R_T along the locus as an "identification label" for the corresponding continuous set of targets. Thus the performance quantities for this target locus can be expressed as functions of R_T instead of $\Delta\phi$, $\Delta\lambda$. Various definitions of target loci can be utilized, e.g., the intersection of the reference ellipsoid with a geocentric plane from the launch-site at a specified azimuth. One method of particular interest is that of specifying the initial trajectory azimuth, α_0 . For any launch-site/target combination there will exist an associated trajectory with some particular α_0 determined uniquely by the specific trajectory calculation process applicable to the system under consideration. Thus we can express α_0 as a function by

$$\alpha_0 = \alpha_0 \{ \phi_L, \Delta\phi, \Delta\lambda, H_L, H_T \} \tag{1.5}$$

Then inversely, $\Delta\phi$, $\Delta\lambda$ can be determined from (1.4), (1.5) if $\phi_L, \alpha_0, R_T, H_L, H_T$ are given, so that

$$\Delta\phi = \Delta\phi \{ \phi_L, \alpha_0, R_T, H_L, H_T \} \tag{1.6}$$

$$\Delta\lambda = \Delta\lambda \{ \phi_L, \alpha_0, R_T, H_L, H_T \} \tag{1.7}$$

Substituting (1.6), (1.7) into (1.1) we can then consider an alternate function for P_P as

$$P_P = P_P \{ \phi_L, \alpha_0, R_T, H_L, H_T \} \tag{1.8}$$

Thus, for a given launch site and target altitude H_T , the performance quantity P_T can be represented by constant probability contours on the surface of the earth as given by (1.1); or alternatively, it can be given, by (1.8) as a function of R_T for various allowable values of α_0 .

2. PROPELLANT RESERVE FUNCTION

The significant performance probability function $\mathcal{P}_p$ discussed in Section 1 can be analyzed conveniently as discussed below by introducing the concept of reserve propellant. For a particular launch-site/target combination we consider a conceptually large population of flight trials without equipment malfunction. For any such flight trial with a normal guidance termination without propellant depletion, let M_T denote the available propellant remaining at final cutoff. For a flight trial with propellant depletion prior to a normal guidance termination we define the corresponding available propellant at burnout as negative. It is important to note that the positive available propellant M_T in the case of a bipropellant liquid engine will not all be burnable, due to outage. By outage is meant the amount of one propellant remaining above its minimum available level when the available mass for the other has been expended. To account for this we define an outage X_o for the final stage by extrapolating the mixture ratio to propellant depletion in some appropriate fashion. In addition, we let $\mathcal{M}_p$ denote the corresponding "propellant reserve" at cutoff — that is, the remaining propellant that could have been burned if guidance termination had been eliminated. Thus,

$$M_T = \mathcal{M}_p + X_o \tag{2.1}$$

Then the probability $\mathcal{P}_p$ of achieving a normal guidance shut down is given by

$\mathcal{P}_p$ = fractional number of trials with a normal guidance termination without propellant depletion

= Probability that $\mathcal{M}_p \geqslant 0$ (2.2)

The quantity $\mathcal{M}_p$ varies with each trial due to variations in the many statistical parameters of the system as well as external disturbing quantities associated with the flight. We assume that in general we can define appropriate quantities $X_1, \cdots X_n$ to represent all such uncertainty effects. A function M_T is then defined by appropriate trajectory calculations and can be represented by

$$M_T = \mathcal{M}_T\{\phi_L, \Delta\phi, \Delta\lambda, H_L, H_T; X_1, \cdots X_n\} \tag{2.3}$$

It is clear that (2.3) is a function of the launch-site/target combination. Recalling (1.6), (1.7) we can consider an alternate function for M_T as

$$M_T = M_T\{\phi_L, \alpha_0, R_T, H_L, H_T; X_1, \cdots X_n\} \tag{2.4}$$

and from (2.1),

$$\mathcal{M}_p = M_T\{\phi_L, \alpha_0, R_T, H_L, H_T; X_1, \cdots X_n\} - X_0 \tag{2.5}$$

If we know the statistics of $X_0, \cdots X_n$, then utilizing the function (2.5), it is in principle straight forward to calculate $f_p(\mathcal{M}_p)$, the probability distribution function for $\mathcal{M}_p$, e.g. by a Monte Carlo computing technique. Thus from (2.2) we see that $\mathcal{P}_p$ can be evaluated as

$$\mathcal{P}_p = \int_0^{\infty} f_p(\mathcal{M}_p)\, d\mathcal{M}_p \tag{2.6}$$

A practical approximation for the calculation of $\mathcal{P}_p$ involves a linear (power series) expansion of the function (2.4) about a set of nominal values. Let X_{KN} denote a conveniently chosen nominal value of X_K, with $K = 0, 1, \cdots n$ such that

$$X_K = X_{KN} + \Delta X_K \tag{2.7}$$

Also, let

$$H_L = H_{LN} + \Delta H_L \tag{2.8}$$

$$H_T = H_{TN} + \Delta H_T \tag{2.9}$$

If the Δ's are small, then we write (2.5) as

$$\mathcal{M}_p = \mathcal{M}_{PN} + \Delta M_H + U \tag{2.10}$$

where $M_{TN} = M_{TN}(\phi_L, \alpha_o, R_T) \equiv M_T\{\phi_L, \alpha_o, R_T, H_{LN}, H_{TN}; X_{1N}, \cdots X_{nN}\}$ (2.11)

$$\mathcal{M}_{PN} = \mathcal{M}_{PN}(\phi_L, \alpha_o, R_T) \equiv M_{TN} - X_{ON} \tag{2.12}$$

$$\Delta M_H \equiv B_{HL}\,\Delta H_L + B_{HT}\,\Delta H_T \tag{2.13}$$

$$B_{HL} = B_{HL}(\phi_L, \alpha_o, R_T) \equiv \frac{\partial M_T}{\partial H_{LN}}\{\phi_L, \alpha_o, R_T, H_{LN}, H_{TN}; X_{1N}, \cdots X_{nN}\} \tag{2.14}$$

$$B_{HT} = B_{HT}(\phi_L, \alpha_o, R_T) \equiv \frac{\partial M_T}{\partial H_{TN}}\{\phi_L, \alpha_o, R_T, H_{LN}, H_{TN}; X_{1N}, \cdots X_{nN}\} \tag{2.15}$$

and

$$U \equiv \sum_{K=0}^{n} B_K\,\Delta X_K \tag{2.16}$$

$$B_0 \equiv -1;\; B_K = B_K(\phi_L, \alpha_o, R_T) \equiv \frac{\partial M_T}{\partial X_{KN}}\{\phi_L, \alpha_o, R_T, H_{LN}, H_{TN}; X_{1N}, \cdots X_{nN}\} \quad (K \geq 1) \tag{2.17}$$

Then letting $\overline{\mathcal{M}}_p$ denote the expected value of $\mathcal{M}_p$, we write

$$\begin{aligned}\overline{\mathcal{M}}_p = \overline{\mathcal{M}}_p(\phi_L, \alpha_o, R_T, H_L, H_T) &= \mathcal{M}_{PN} + \Delta M_H + \overline{U} \\ &= M_{TN} + B_{HL}(H_L - H_{LN}) + B_{HT}(H_T - H_{TN}) + \sum_{K=1}^{n} B_K(\overline{X}_K - X_{KN}) - \overline{X}_0\end{aligned} \tag{2.18}$$

Thus (2.10) becomes

$$\begin{aligned}\mathcal{M}_p &= \overline{\mathcal{M}}_p + U - \overline{U} \\ &= \overline{\mathcal{M}}_p + \sum_{K=0}^{n} B_K(X_K - \overline{X}_K)\end{aligned} \tag{2.19}$$

The standard deviation σ_p for the statistical quantity $\mathcal{M}_p$ is given by

$$\sigma_p = \left\{ \overline{(\mathcal{M}_p - \overline{\mathcal{M}}_p)^2} \right\}^{1/2}$$

$$= \left\{ \overline{\left[\sum_{k=0}^{n} B_k (x_k - \bar{x}_k) \right]^2} \right\}^{1/2}$$

$$= \left\{ \sum_{k=0}^{n} \sum_{i=0}^{n} B_k B_i \rho_{ik} \sigma_i \sigma_k \right\}^{1/2}$$

$$= \left\{ \sum_{k=0}^{n} B_k^2 \sigma_k^2 + \sum_{k=1}^{n} \sum_{i=0}^{k-1} 2 \rho_{ki} B_k B_i \sigma_k \sigma_i \right\}^{1/2} \qquad (2.20)$$

where

$$\sigma_k \equiv \left\{ \overline{(x_k - \bar{x}_k)^2} \right\}^{1/2} \qquad (2.21)$$

$$\rho_{ik} = \rho_{ki} \equiv \frac{1}{\sigma_k \sigma_i} \overline{(x_k - \bar{x}_k)(x_i - \bar{x}_i)} \qquad (2.22)$$

Let us approximate the probability distribution function $f_p(\mathcal{M}_p)$ by a normal distribution having the same mean $\overline{\mathcal{M}}_p$ and standard deviation σ_p as $\mathcal{M}_p$. Then (2.6) can be approximated as

$$\mathcal{P}_p = P_p\{\phi_L, A_0, R_T, H_L, H_T\} = \int_{u=0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-n_p)^2} du$$

$$= \int_{u=-n_p}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= \frac{1}{2} + \int_0^{n_p} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \qquad (2.23)$$

where

$$n_p = n_p\{\phi_L, \alpha_0, R_T, H_L, H_T\} = \overline{\mathcal{M}}_p / \sigma_p \qquad (2.24)$$

Thus the quantity $\mathcal{P}_p$ depends only on the value of n_p for the launch-site/target combination of interest. From (2.24) and recalling (2.18), (2.20) we see that the following quantities are involved:

(1) **Selected nominal values:**

$$H_{LN}, H_{TN}, X_{0N}, \cdots X_{nN}$$

(2) **System statistical quantities:**

$$\bar{X}_0, \cdots \bar{X}_n$$

$$\sigma_0, \cdots \sigma_n$$

$$\rho_{ik} \quad (i \neq k)$$

(3) Trajectory quantities dependent on ϕ_L, α_0, R_T :

$$M_{TN}$$

$$B_{H_L}, B_{HT}, B_1, \cdots B_n$$

3. RANGE FUNCTION

Instead of the function M_T discussed in Sec. 2 it is sometimes useful to formulate the performance analysis in terms of a range function as follows. For a particular launch-site/target combination we determine all trajectory-related parameters by the applicable targeting calculation process for the system. Then we consider the hypothetical situation such that the missile operates under closed-loop guidance, except that instead of guidance termination we cutoff thrust when the available propellant reaches some arbitrarily selected value M_{bo}, where M_{bo} includes the extrapolated outage and reserve propellant. If the system utilizes a vernier we include a nominal vernier phase. The resulting range $\mathcal{R}$ is thus a function defined by appropriate trajectory calculations and can be expressed as

$$\mathcal{R} = \mathcal{R}\{\phi_L, \Delta\phi, \Delta\lambda, H_L, H_T; X_1, \cdots X_n, M_{bo}\} \tag{3.1}$$

Recalling (1.6), (1.7) we can consider an alternate function for $\mathcal{R}$ as

$$\mathcal{R} = R\{\phi_L, \alpha_o, R_T, H_L, H_T; X_1, \cdots X_n, M_{bo}\} \tag{3.2}$$

In general the functional form of $\mathcal{R}$ depends on the launch-site/target combination as indicated in (3.2), due to the trajectory shaping and guidance steering that is dependent on this combination. However, for given $\phi_L, \alpha_o, H_L, H_T$ some systems utilize the same trajectory shaping regardless of R_T, and we can write a corresponding function for $\mathcal{R}$ as

$$\mathcal{R} = \mathcal{R}\{\phi_L, \alpha_0, H_L, H_T ; X_1, \cdots X_n, M_{bo}\} \qquad (3.3)$$

In such a case, $\mathcal{R}$ as a function of M_{bo} for particular values of ϕ_L, α_0, etc., is the same as range to the instantaneous impact point versus available propellant for a single trajectory calculation. For systems in which this is not the case, $\mathcal{R}$ must be obtained by a separate trajectory calculation for a particular value of R_T by utilizing the corresponding targeting calculation to secure the proper associated trajectory shaping.

To obtain the relation between the range function R and the function M_T of (2.4) we recall that $M_T = M_{bo}$ when $\mathcal{R} = R_T$. Thus M_T is defined implicitly as a function of $\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n$ by

$$R_T = R\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\} \qquad (3.4)$$

Introducing the nominal quantities $X_{1N}, \cdots X_{nN}$ and recalling (2.11) we obtain

$$R_T = R\{\phi_L, \alpha_0, R_T, H_{LN}, H_{TN}, X_{1N}, \cdots X_{nN}, M_{TN}\} \qquad (3.5)$$

Thus (3.5) determines M_{TN} implicitly. To calculate n_p we require the additional trajectory quantities B_{H_L}, B_{H_T} defined by (2.14), (2.15) and $B_1, \cdots B_n$ as defined by (2.17). Thus we write (3.4) in differential form, with ϕ_L, α_0, R_T held constant:

$$0 = \frac{\partial R\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\}}{\partial H_L} dH_L + \frac{\partial R\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots M_T\}}{\partial H_T} dH_T$$
$$+ \frac{\partial R\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\}}{\partial M_T} dM_T + \sum_{k=1}^{n} \frac{\partial R\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots M_T\}}{\partial X_k} dX_k \qquad (3.6)$$

Therefore,

$$\frac{\partial M_T}{\partial X_K}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1 \cdots X_n\} = - \frac{\frac{\partial R}{\partial X_K}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\}}{\frac{\partial R}{\partial M_T}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\}} \quad (3.7)$$

$$\frac{\partial M_T}{\partial H_L}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n\} = - \frac{\frac{\partial R}{\partial H_L}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\}}{\frac{\partial R}{\partial M_T}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\}} \quad (3.8)$$

$$\frac{\partial M_T}{\partial H_T}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n\} = - \frac{\frac{\partial R}{\partial H_T}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\}}{\frac{\partial R}{\partial M_T}\{\phi_L, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, M_T\}} \quad (3.9)$$

Let

$$A_0 = A_0(\phi_L, \alpha_0, R_T) = \frac{\partial R}{\partial M_{TN}}\{\phi_L, \alpha_0, R_T, H_{LN}, H_{TN}, X_{1N}, \cdots X_{nN}, M_{TN}\} \quad (3.10)$$

$$A_{HL} = A_{HL}(\phi_L, \alpha_0, R_T) = \frac{\partial R}{\partial H_{LN}}\{\phi_L, \alpha_0, R_T, H_{LN}, H_{TN}, X_{1N}, \cdots X_{nN}, M_{TN}\} \quad (3.11)$$

$$A_{HT} = A_{HT}(\phi_L, \alpha_0, R_T) = \frac{\partial R}{\partial H_{TN}}\{\phi_L, \alpha_0, R_T, H_{LN}, H_{TN}, X_{1N}, \cdots X_{nN}, M_{TN}\} \quad (3.12)$$

and for $k = 1, \cdots n$:

$$A_k = A_k(\phi_L, \alpha_0, R_T) = \frac{\partial R}{\partial X_{kN}}\{\phi_L, \alpha_0, R_T, H_{LN}, H_{TN}, X_{1N}, \cdots X_{nN}, M_{TN}\} \quad (3.13)$$

Then recalling (2.14), (2.15), (2.17) we obtain the following important results:

$$B_{HL} = - \frac{A_{HL}}{A_0} \quad (3.14)$$

$$B_{HT} = - \frac{A_{HT}}{A_0} \quad (3.15)$$

for $K = 0, \cdots n$

$$B_k = -\frac{A_k}{A_0} \tag{3.16}$$

where we expect $A_0 < 0$.

With M_{TN} and $B_{H_L}, B_{H_T}, B_1, \cdots B_n$ determined we can write the linear function $\mathcal{M}_p$ given by (2.10) and can calculate $\mathcal{R}_p$ from (2.18), (2.20, (2.23), (2.24).

It is useful to approximate the range function (3.2) by a linear expansion about $H_{LN}, H_{TN}, X_{1N}, \cdots X_{nN}, M_{TN}$. Thus from (3.2), (3.5) we write

$$\mathcal{R} = R_T + A_{H_L}(H_L - H_{LN}) + A_{H_T}(H_T - H_{TN}) + \sum_{k=1}^{n} A_k(X_k - X_{kN}) + A_0(M_{b0} - M_{TN}) \tag{3.17}$$

A special case for the function $\mathcal{R}$ that is important in studying the effect of propellant depletion on impact statistics corresponds to missile operation until propellant depletion. Thus we have $M_{b0} = X_0$ = outage for the final stage. The corresponding range function is denoted by $\mathcal{R}_p$ and represented by

$$\mathcal{R}_p = R\{A, \alpha_0, R_T, H_L, H_T, X_1, \cdots X_n, X_0\} \tag{3.18}$$

The linearized form for $\mathcal{R}_p$ is given by (2.12), (3.17) as

$$\mathcal{R}_p = R_T + \Delta R_H + Y - A_0 \mathcal{M}_{pN} \tag{3.19}$$

where, utilizing (2.15), (3.14), (3.15),

$$\Delta R_H = A_{HL}(H_L - H_{LN}) + A_{HT}(H_T - H_{TN})$$

$$= - A_0 \Delta M_H \qquad (3.20)$$

and utilizing (2.16), (3.16)

$$Y \doteq \sum_{k=0}^{n} A_k(\gamma_k - \gamma_{kN})$$

$$= - A_0 U \qquad (3.21)$$

Recalling (2.10), then (3.19) can be written as

$$\mathcal{R}_p = R_T - A_0 \mathcal{M}_p \qquad (3.22)$$

Then

$$\bar{\mathcal{R}}_p = R_T + \Delta R_H + \bar{Y} - A_0 \mathcal{M}_{pN} \qquad (3.23a)$$

$$= R_T - A_0 \bar{\mathcal{M}}_p \qquad (3.23b)$$

Thus

$$\mathcal{R}_p = \bar{\mathcal{R}}_p - A_0(\mathcal{M}_p - \bar{\mathcal{M}}_p) \qquad (3.24)$$

Recalling (2.20), we obtain the standard deviation σ_R for $\mathcal{R}_p$ as

$$\sigma_R = \left\{ \overline{(\mathcal{R}_p - \bar{\mathcal{R}}_p)^2} \right\}^{1/2}$$

$$= |A_0| \, \sigma_p \qquad (3.25)$$

We therefore approximate the probability distribution function $f_R(\mathcal{R}_p)$ for $\mathcal{R}_p$ by

$$f_R(\mathcal{R}_p) = \frac{1}{\sqrt{2\pi}\,\sigma_R} \, e^{-\{\mathcal{R}_p - \bar{\mathcal{R}}_p\}^2 / 2\sigma_R^2} \qquad (3.26)$$

The cumulative probability $\mathcal{P}_{R_p \geq R_o}$ is given by

$$\mathcal{P}_{R_p \geq R_o} = \int_{R_p = R_o}^{\infty} f_R(R_p)\, dR_p$$

$$= \frac{1}{2} + \int_0^{n_p + (R_T - R_o)/\sigma_R} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \qquad (3.27)$$

where $n_p = n_p(\phi_L, \alpha_o, R_T, h_L, H_T)$ and $\sigma_R = \sigma_R(\phi_L, \alpha_o, R_T, H_L, H_T)$

We recall that R_p relates to the hypothetical situation of a trajectory corresponding to a target at R_T, for which we eliminate guidance termination and continue to propellant depletion; and $\mathcal{P}_{R_p \geq R_o}$ is the probability that in such a situation a range of at least R_o will be achieved. Thus $\mathcal{P}_{R_p \geq R_T}$ is identical with $\mathcal{P}_p$, which can be seen from (3.27) by comparing with (2.23).

4. ADDITIONAL PERFORMANCE CALCULATIONS

It is often of interest to consider target-range for given ϕ_L, α_0 corresponding to some specified value of the probability $\mathcal{P}_p$. In particular, comparison between various systems is more meaningful in terms of differences in target-range at the same probability level rather than differences in probability for the same target-range. Thus for any change in parameter values we are interested in the target-range $R_T^\dagger$ to maintain the same probability value $\mathcal{P}_p$ corresponding to R_T for the previous conditions. An important example of how this calculation can be accomplished is given below in discussing the method of handling launch-site and target altitude quantities differing from standard values.

Launch-Site and Target Altitude

Let us suppose we have obtained the probability results for the nominal conditions $H_L = H_{LN} = H_L^*$, $H_T = H_{TN} = H_T^*$ where H_L^*, H_T^* are convenient standard values chosen for suitable linear expansion. Denoting this special result as $\mathcal{P}_p = \mathcal{P}_p^*(\phi_L, \alpha_0, R_T) \equiv \mathcal{P}_p(\phi_L, \alpha_0, R_T, H_L^*, H_T^*)$ (4.1)

we write from (2.18), (2.23), (2.24):

$$\mathcal{P}_p^* = \frac{1}{2} + \int_0^{n_p^*} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \, du \qquad (4.2)$$

and

$$n_p^* = n_p^*(\phi_L, \alpha_0, R_T) \equiv n_p(\phi_L, \alpha_0, R_T, H_L^*, H_T^*)$$

where

$$n_p^* = \frac{1}{\sigma_p}\left\{ M_{TN}^* + \sum_{K=1}^{n} B_K^*(\phi_L, \alpha_o, R_T)[\bar{X}_K - X_{KN}] - \bar{X}_o \right\} \tag{4.3}$$

and

$$M_{TN}^* = M_{TN}^*(\phi_L, \alpha_o, R_T) \equiv M_T(\phi_L, \alpha_o, R_T, H_L^*, H_T^*; X_{1N}, \cdots X_{nN}) \tag{4.4}$$

We seek to determine the value of $R_T^\dagger$ corresponding to $P_p = P_p^*$ with $H_L = H_L^* + \Delta H_L$, $H_T = H_T^* + \Delta H_T$. Thus the basic condition for determining $R_T^\dagger$ is given as follows:

$$P_p^* = P_p(\phi_L, \alpha_o, R_T, H_L^*, H_T^*) = P_p(\phi_L, \alpha_o, R_T^\dagger, H_L, H_N)$$
$$= \frac{1}{2} + \int_0^{n_p^\dagger} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \tag{4.5}$$

where

$$n_p^\dagger = \frac{1}{\sigma_p}\Big\{ M_{TN}^\dagger + B_{HL}^*(H_L - H_L^*) + B_{HT}^*(H_T - H_T^*)$$
$$+ \sum_{K=1}^{n} B_K^*(\phi_L, \alpha_o, R_T^\dagger)(\bar{X}_K - X_{KN}) - \bar{X}_o \Big\} \tag{4.6}$$

and

$$M_{TN}^\dagger \equiv M_{TN}^*(\phi_L, \alpha_o, R_T^\dagger) = M_T(\phi_L, \alpha_o, R_T^\dagger, H_L^*, H_T^*; X_{1N} \cdots X_{nN}) \tag{4.7}$$

Thus

$$n_p^\dagger = n_p^* \tag{4.3}$$

As $R_T \approx R_T^\dagger$, then in keeping with the linearization assumption being utilized,

$$B_K^* = B_K^*(\phi_L, \alpha_o, R_T) \equiv \frac{\partial M_T(\phi_L, \alpha_o, R_T, H_L^*, H_T^*, X_{1N}, \cdots X_{nN})}{\partial X_{KN}}$$
$$\approx B_K^*(\phi_L, \alpha_o, R_T^\dagger) \tag{4.9}$$

Then substituting (4.3), (4.6) into (4.8) and utilizing (4.9), we obtain

$$B^*_{HL}(H_L - H^*_L) + B^*_{HT}(H_T - H^*_T) = M^*_{TN} - M^\dagger_{TN} \qquad (4.10)$$

Recalling (3.5), (3.10) we write

$$\begin{aligned} R^\dagger_T &= R\{\phi_L, \alpha_o, R^\dagger_T, H^*_L, H^*_T; x_{1N}, \cdots x_{nN}, M^\dagger_{TN}\} \\ &\simeq R\{\phi_L, \alpha_o, R_T, H^*_L, H^*_T; x_{1N}, \cdots x_{nN}, M^\dagger_{TN}\} \\ &= R\{\phi_L, \alpha_o, R_T, H^*_L, H^*_T; x_{1N}, \cdots x_{nN}, M^*_{TN}\} \\ &\quad + \frac{\partial R}{\partial M^*_{TN}}\{\phi_L, \alpha_o, R_T, H^*_L, H^*_T, x_{1N}, \cdots x_{nN}, M^*_{TN}\}[M^\dagger_{TN} - M^*_{TN}] \\ &= R_T + A^*_o(M^\dagger_{TN} - M^*_{TN}) \qquad (4.11) \end{aligned}$$

In step two above we have neglected the change in the <u>form</u> of the range function due to the retargeting from R_T to $R^\dagger_T$ as in (3.3). Combining (4.10), (4.11) and utilizing (3.14), (3.15) we obtain

$$R^\dagger_T = R_T + \Delta R^*_H \qquad (4.12)$$

where as in (3.20),

$$\Delta R^*_H = A^*_{HL}(H_L - H^*_L) + A^*_{HT}(H_T - H^*_T) \qquad (4.13)$$

Thus if R_T is the target-range for a probability ρ^*_p determined for standard altitudes H^*_L, H^*_T, then we add ΔR^*_H to it to get the target-range at the same probability ρ^*_p corresponding to altitudes H_L, H_T.

Changes in Expected Values

Another performance calculation of interest is that arising from a re-evaluation of the expected values $\bar{x}_1, \cdots \bar{x}_n$ for which we have previously obtained the result $\mathcal{P}_p = P_p(\phi_L, \alpha_0, R_T, H_L, H_T)$. Let the new expected values be denoted by $\bar{X}_0^{\dagger}, \ldots \bar{X}_n^{\dagger}$ and let the standard deviations $\sigma_0, \cdots \sigma_n$ remain unchanged. We can obtain the new probability $\mathcal{P}_p^{\dagger} = \mathcal{P}_p^{\dagger}(\phi_L, \alpha_0, R_T, H_L, H_T)$ as follows. Now the nominal expansion (2.10) is independent of the expected values. Thus from (2.18), (2.24) and assuming this linear approximation to be valid over the interval including the new expected values, with dispersion around them, we write

$$n_p^{\dagger} = \frac{1}{\sigma_p}\left\{\mathcal{K}_{pN} + \Delta M_H + \sum_{k=0}^{n} B_k(\bar{X}_k^{\dagger} - X_{kN})\right\}$$

$$= n_p + \Delta n_p \qquad (4.14)$$

where $n_p = n_p(\phi_L, \alpha_0, R_T, H_L, H_T)$ corresponds to $(\bar{X}_0, \cdots \bar{X}_n)$,

and

$$\Delta n_p \equiv \frac{1}{\sigma_p}\sum_{k=0}^{n} B_k(\bar{X}_k^{\dagger} - \bar{X}_k) \qquad (4.15a)$$

$$= -\frac{A_d}{A_0}\frac{1}{\sigma_R}\sum_{k=0}^{n} A_k(\bar{X}_k^{\dagger} - \bar{X}_k) \qquad (4.15b)$$

Then from (2.23),

$$P_p^{\dagger}(\phi_L, \alpha_0, R_T, H_L, H_T) = \frac{1}{2} + \int_0^{n_p + \Delta n_p} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

$$= P_p(\phi_L, \alpha_0, R_T, H_L, H_T) + \Delta\mathcal{P}_p \qquad (4.16)$$

where

$$\Delta\mathcal{P}_p \equiv \int_{n_p}^{n_p + \Delta n_p} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \qquad (4.17)$$

As discussed previously, instead of calculating a new value $\mathcal{P}_p^{\dagger}$ for a particular launch-site/target combination $(\phi_L, \alpha_0, R_T, H_L, H_T)$, we may be interested in determining the target-range increment at some particular value of probability arising from the change in expected values. That is, we seek to determine $R_T^\dagger$ such that

$$P_p(\phi_L, \alpha_0, R_T, H_L, H_T) = P_p^\dagger(\phi_L, \alpha_0, R_T^\dagger, H_L, H_T) \tag{4.18}$$

From (2.23) we see that

$$n_p(\phi_L, \alpha_0, R_T, H_L, H_T) = n_p(\phi_L, \alpha_0, R_T^\dagger, H_L, H_T) \tag{4.19}$$

From (2.24), (2.18) and assuming $B_k = B_k(\phi_L, \alpha_0, R_T) \approx B_k(\phi_L, \alpha_0, R_T^\dagger)$ as in (4.9), then we obtain

$$\begin{aligned} M_{TN}^\dagger - M_{TN} &= \sum_{k=0}^{n} B_k(\bar{X}_k - \bar{X}_k^\dagger) \\ &= \frac{1}{A_0}\sum_{k=0}^{n} A_k(\bar{X}_k^\dagger - \bar{X}_k) \end{aligned} \tag{4.20}$$

where we have utilized (3.16) and

$$M_{TN} = M_{TN}(\phi_L, \alpha_0, R_T) \tag{4.21}$$

$$M_{TN}^\dagger = M_{TN}(\phi_L, \alpha_0, R_T^\dagger) \tag{4.22}$$

Recalling (3.5), then as in (4.11), we obtain

$$R_T^\dagger = R_T + A_0(M_{TN}^\dagger - M_{TN}) \tag{4.23}$$

Then substituting (4.20) into (4.23) we obtain

$$R_T^\dagger = R_T + \Delta R^\dagger \tag{4.24}$$

where

$$\Delta R^\dagger \equiv \sum_{k=0}^{n} A_k(\bar{x}_k^\dagger - \bar{x}_k) \tag{4.25a}$$

$$= -\frac{|A_d|}{A_0}\, \Delta n_p\, \sigma_R \tag{4.25b}$$

with Δn_p given by (4.15) and σ_R by (3.25).

The expressions (4.25a) and (4.13) show the adjustments in target-range under parameter changes, at constant probability. Accordingly, we say that range is exchanged for a compensating parameter change and call the quantities $A_{H_L}, A_{H_T}, A_o, \cdots A_n$ "range exchange coefficients." For the "maximum target-range" case discussed in Sec. 1 they would be called specifically "maximum target-range exchange coefficients" , and it is this set of exchange coefficients that we are most interested in.

Additional Approximations

A useful approximation that has been found valid for some systems is to consider the quantity σ_p of (2.20) as constant and independent of the launch-site/target combination. Thus with σ_p determined in advance it only remains to find $\overline{\mathcal{M}}_p$ in order to calculate $\mathcal{P}_p$ for any $\phi_L, \alpha_o, R_T, H_L, H_T$. If $H_L = H_{LN}$, $H_T = H_{TN}$ and the nominal quantities $x_{1N}, \cdots x_{nN}$ are selected as expected values we see from (2.18) that M_{TN} is the only trajectory quantity required to determine $\mathcal{P}_p$ for the target of interest. This is particularly useful for determining $\mathcal{P}_p$ as part of an operational targeting calculation.

For the special case of $H_L = H_{LN}$, $H_T = H_{TN}$ for all ranges and for given ϕ_L, α_o , then the probability function $\mathcal{P}_p$ is determined by $\overline{\mathcal{M}}_p$ which can be represented by a function

$$\begin{aligned} \overline{\mathcal{M}}_p &= M_{TN}(\phi_L, \alpha_o, R_T) - \overline{Y}_o \\ &= \overline{\mathcal{M}}_p(R_T) \end{aligned} \tag{4.26}$$

To differentiate (4.26) we recall (3.5), (3.10) and obtain

$$1 = \frac{\partial R}{\partial M_{TN}}\{\phi_L, \alpha_0, R_T, H_{LN}, H_{TN}, x_{1N}, \cdots x_{nN}, M_{TN}\} \frac{\partial M_{TN}(\phi_L, \alpha_0, R_T)}{\partial R_T}$$

$$= A_0 \frac{\partial \overline{\mathcal{M}}_p(R_T)}{\partial R_T} \tag{4.27}$$

or

$$\frac{\partial \overline{\mathcal{M}}_p(R_T)}{\partial R_T} = \frac{1}{A_0} \tag{4.28}$$

In (4.27) we have neglected the change in form of the function $\mathcal{R}$ for different targets, as in (3.3).

The function given by (4.26) completely determines σ_p as a function of R_T. As $\overline{\mathcal{M}}_p(R_T)$ varies almost linearly with R_T, not many determinations for values of R_T are required. This suggests an approximation in which we introduce linear extrapolation by a determination of the function and its slope at some range $\widetilde{R}_T$. This would be effected over the interval of all ranges of interest. We note that this differs somewhat from the approximation (2.10) in which we linearize only over the dispersion for a given target. To discuss such a procedure we first define

$$\widetilde{A}_0 \equiv A_0(\phi_L, \alpha_0, \widetilde{R}_T) \tag{4.29}$$

$$\widetilde{\sigma}_R \equiv \sigma_R(\phi_L, \alpha_0, \widetilde{R}_T) = |\widetilde{A}_0| \, \sigma_p \tag{4.30}$$

$$\widetilde{\mathcal{M}}_{pN} = \overline{\mathcal{M}}_p(\widetilde{R}_T) \equiv M_{TN}(\phi_L, \alpha_0, \widetilde{R}_T) - \overline{X}_0 \tag{4.31}$$

A linear approximation to (4.26) can be written as

$$\overline{\mathcal{M}}_p(R_T) = \overline{\mathcal{M}}_p^{\dagger}(R_T) \equiv \overline{\mathcal{M}}_p(\widetilde{R}_T) + \frac{\partial \overline{\mathcal{M}}_p(\widetilde{R}_T)}{\partial \widetilde{R}_T}[R_T - \widetilde{R}_T]$$

$$= \widetilde{\mathcal{M}}_{pN} + \frac{1}{\widetilde{A}_0}[R_T - \widetilde{R}_T] \tag{4.32}$$

Utilizing (2.24), we obtain

$$n_p^\dagger = n_p^\dagger(R_T) = \frac{-|\tilde{A}_0|}{\tilde{A}_0}\left\{\frac{\tilde{R}_0 - R_T}{\tilde{\sigma}_R}\right\} \qquad (4.33)$$

where

$$\tilde{R}_0 \equiv \tilde{R}_T - \tilde{A}_0 \tilde{\mathcal{M}}_{pN} \qquad (4.34)$$

Thus from (2.23),

$$\mathcal{P}_p = \frac{1}{2} + \int_0^{-\frac{|A_0|}{A_0}\left\{\frac{\tilde{R}_0 - R_T}{\tilde{\sigma}_R}\right\}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \qquad (4.35)$$

We note that $\mathcal{P}_p = 0.5$ when $R_T = \tilde{R}_0$ and that the quantities $\tilde{R}_0$, $\tilde{\sigma}_R$ completely determine $\mathcal{P}_p$ in this method of approximation. Thus if the nominals are selected as expected values and if $\tilde{\mathcal{M}}_{pN} = 0$, then from (4.32), $\tilde{R}_0 = \tilde{R}_T$ and only $\tilde{\sigma}_R$ is required to determine $P_p(\phi_R, \alpha_0, R_T)$. Selecting $\tilde{R}_T$ at .50 probability however requires a linear extrapolation over a longer interval than if $\tilde{R}_T$ corresponds to a reasonably high probability, and is therefore not recommended.

To calculate the error due to the approximation (4.32) let

$$\mathcal{P}_p^\dagger = \mathcal{P}_p^\dagger(R_T) = \frac{1}{2} + \int_0^{\frac{1}{\sigma_p}\overline{\mathcal{M}}_p^\dagger(R_T)} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \qquad (4.36)$$

and

$$\mathcal{P}_p = \mathcal{P}_p(R_T) = \frac{1}{2} + \int_0^{\frac{1}{\sigma_p}\overline{\mathcal{M}}_p(R_T)} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \qquad (4.37)$$

To determine the difference between the approximate target-range $R_T^\dagger$ and the more correct value R_T for an arbitrary probability level $\mathcal{P}_p$ we write

$$\mathcal{P}_p(R_T) = \mathcal{P}_p^{\ddagger}(R_T^{\ddagger}) \tag{4.38}$$

or

$$\begin{aligned}\overline{\mathcal{M}}_p(R_T) &= \overline{\mathcal{M}}_p^{\ddagger}(R_T^{\ddagger}) \\ &= \tilde{\mathcal{M}}_{pN} + \frac{1}{\tilde{A}_0}\left[R_T^{\ddagger} - \tilde{R}_T\right] \\ &= \tilde{\mathcal{M}}_{pN} + \frac{1}{\tilde{A}_0}\left[R_T - \tilde{R}_T\right] + \frac{R_T^{\ddagger} - R_T}{\tilde{A}_0}\end{aligned} \tag{4.39}$$

Therefore

$$R_T = R_T^{\ddagger} + \Delta R^{\ddagger} \tag{4.40}$$

where the error $\Delta R^{\ddagger}$ at the probability of interest is given by

$$\Delta R^{\ddagger} \equiv -\tilde{A}_0\left\{\overline{\mathcal{M}}_p(R_T) - \overline{\mathcal{M}}_p^{\ddagger}(R_T)\right\} \tag{4.41}$$

This is shown schematically in Fig. 4.1

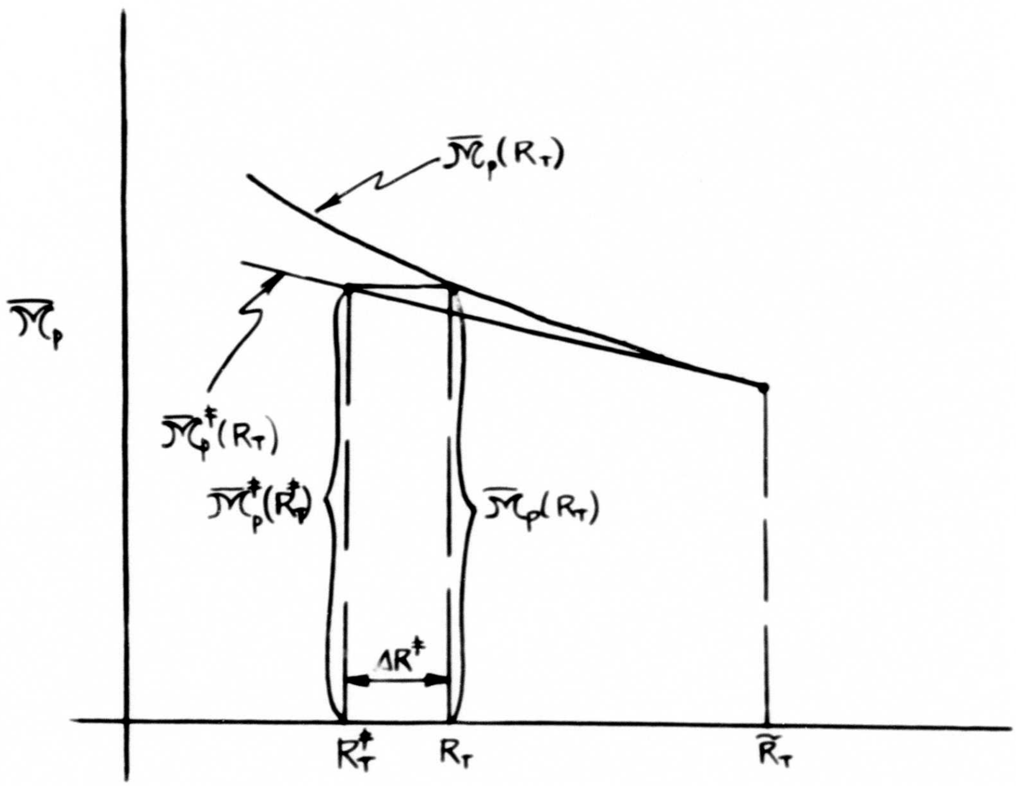

Fig. 4.1

INDEX OF SYMBOLS

Symbol	Equation	Page
A_0	3.10	3.3
A_{ML}	3.11	3.3
A_{MT}	3.12	3.3
A_K	3.13	3.3
α_0	1.5	1.4
B_{ML}	2.14	2.3
B_{MT}	2.15	2.3
B_K	2.17	2.3
B_K^*	4.4	4.2
H_L	-	1.3
ΔH_L	2.8	2.3
H_{LN}	-	2.3
H_L^*	-	4.1
H_T	-	1.3
ΔH_T	2.9	2.3
H_{TN}	-	2.3
H_T^*	-	4.1
λ_L	-	1.3
λ_T	-	1.3
$\Delta\lambda$	1.3	1.3
M_{bo}	-	3.1
M_p	2.1	2.1
$M_p^\dagger$	4.32	4.7
$\bar{M}_p$	2.18	2.3
M_{pN}	2.12	2.3
M_T	2.3	2.1
M_{TN}	2.11	2.3
M_{TN}^*	4.4	4.2
ΔM_N	2.13	2.3
n_p	2.24	2.4
n_p^*	4.3	4.2
$n_p^\dagger$	4.33	4.8
Δn_p	4.15	4.4
$\mathcal{P}_p$	-	1.2

Symbol	Equation	Page
$\mathcal{P}_p^*$	4.2	4.1
$\mathcal{P}_p^\dagger$	4.17	4.4
$\mathcal{R}$	3.1	3.1
	3.2	3.1
	3.3	3.2
	3.17	3.4
$\mathcal{P}_{,p}$	3.18	3.4
	3.19	3.4
	3.24	3.5
$\bar{\mathcal{R}}_p$	3.23	3.5
R_T	1.4	1.3
$\Delta R^\dagger$	4.25	4.5
$\Delta R^\dagger$	4.41	4.9
ΔR_N	3.20	3.5
ΔR^*	4.13	4.3
ρ_{iN}	2.22	2.4
σ_K	2.21	2.4
σ_p	2.20	2.4
σ_R	3.25	3.5
U	2.16	2.3
φ_L	-	1.3
φ_T	-	1.3
$\Delta\varphi$	1.2	1.3
X_0	-	2.1
X_K	-	2.2
X_{KN}	-	2.3
ΔX_K	2.7	2.3
Y	3.21	3.5

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Note: Additional documents are listed in Reference 2.

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